78-469

Comparison of Optimality Criteria Algorithms for Minimum Weight Design of Structures

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This paper presents a comparison of frequently used optimization algorithms based on optimality criteria to design a minimum weight structure. After summarizing the different methods, the relationship between the various algorithms is shown. They differ only in the degree of approximations made in formulating the recurrence relations to modify the design variables and to evaluate the Lagrange multipliers. A new iterative scheme, similar to Newton-Raphson, is also presented, with the equations written in such a form that it is not necessary to select the initial design vector of the unknown Lagrange multipliers. It is shown that with this scheme a minimum weight design can be obtained with a smaller number of analyses of the structure than with previously proposed methods.

I. Introduction

THE objective in structural optimization is to design a minimum weight structure that will satisfy all the specified constraints. The constraints may include minimum and maximum sizes, maximum allowable stresses, limitations on the displacements at various points in the structure, buckling loads, dynamic stiffness, frequency requirements, etc. Methods for the optimum design of structures have progressed rapidly in recent years. In particular, optimality criteria approaches have significantly advanced the state-of-the-art of the minimum weight design of structures involving large finite-element assemblies. Optimality criteria and the necessary iterative algorithms were derived and are being used successfully for the optimization of large practical structures with static, dynamic, and stability requirements.[1-20]

The optimization algorithms discussed here consist of two main steps. The first step is to analyze the structure by the finite-element method to determine its response to the applied loads. The second step is to redistribute the material in the members so that the weight of the structure is reduced. In optimality criteria methods, the second step is carried out by using a recurrence relation derived from the appropriate optimality criterion. In any procedure, the algorithm must answer two basic questions in producing a new design step, i.e., in what direction to go and how far. There are no clear instructions obtainable from any theory that give a unique answer to these two simple questions. Numerous algorithms have been proposed by various investigators and it appears productive at this time to show how they are related to each other, i.e., in what sense they are basically identical and in what important and unimportant details they are different.

The recurrence relation for the design variables is derived from the optimality criterion and contain unknown Lagrange multipliers that correspond to the constraints. The multipliers must be evaluated before the recurrence relations for the variables can be used. The numerical procedures to evaluate the Lagrange multipliers are derived using the constraint equations. Optimality criteria methods may differ for the same problem both in the recurrence relations for the variables and in the numerical procedure to evaluate the multipliers. For example, in Refs. 1-14 an exponential recurrence relation is used for the design variables, while a linear relation is derived in Ref. 15 and used in Refs. 16-17. The Newton-Raphson iterative technique is proposed in Ref. 7 to evaluate the Lagrange multipliers. On the other hand, a recurrence relation based on the ratio of the current and limiting values of the constraints is used for the multipliers in Refs. 8, 11, 12, 14, and 18. In Ref. 15, a linear set of simultaneous equations is derived to determine the Lagrange multipliers. This approach is used in Refs. 15, 16, and 18. In this paper, the interrelation of these various approaches will be shown and the effects of their differences demonstrated by examples.

This paper will consider only displacement and stress constraints; however, the basic relations and conclusions are applicable to other types of constraints. Section II gives the basic equations of finite-element analysis and derives the optimality criterion. A short derivation of different recurrence relations for the design variables and the Lagrange multipliers is given in Sec. III. A comparison of the different relations is made in Sec. IV. Illustrative examples are given in Sec. V and the conclusions are presented in Sec. VI.

II. Optimality Criteria

The optimality criterion for the generalized constraints is derived here first, then it is specialized for the displacement constraint problem.

Consider a structure which is discretized into m finite elements. For this structure, the load displacement relation is written as:

$$[K]\{r\} = \{R\} \tag{1}$$

where $\{R\}$ is the applied load vector, $\{r\}$ is the displacement vector, and $[K]$ is the total stiffness matrix of the structure given by:

$$[K] = \sum_{i=1}^{m} \{a\}_i^t [k]_i \{a\}_i \tag{2}$$

In Eq. (2), $[k]_i$ is the stiffness matrix of the ith element, $\{a\}_i$ is the compatibility matrix of the ith element, and $\{a\}_i^t$ is the transpose of $\{a\}_i$.

The weight of the structure $W(A_i)$ is given by

$$W(A_i) = \sum_{i=1}^{m} \rho_i A_i \ell_i \tag{3}$$

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where ρ_i is the mass density and $A_i\ell_i$ is the volume of the element. The design variable is A_i and ℓ_i is a constant that depends on the geometry of the element. The generalized constraints $f_j(A_i)$ imposed on the structure can be written as

$$f_j(A_i)=C_j(A_i)-\bar{C}_j\leq 0 \quad j=1,...,p \tag{4}$$

where $C_j(A_i)$ is the actual value of the jth constraint and $\bar{C}_j$ is its limiting value or desired value. The total number of constraints on the structure is p.

The objective is to minimize $W(A_i)$ subject to the constraints given in Eq. (4). Using Eqs. (3) and (4), the Lagrange function $W(A_i,\lambda_i)$ can be written as

$$W(A_i,\lambda_j)=\sum_{i=1}^{m}\rho_i\ell_iA_i+\sum_{j=1}^{p}\lambda_jf_j(A_i) \tag{5}$$

where λ_j are the Lagrangian parameters. The necessary conditions for the local constrained optimum are obtained by differentiating Eq. (5) with respect to the design variables A_i. This gives

$$\rho_i\ell_i+\sum_{j=1}^{p}\lambda_j\frac{\partial}{\partial A_i}f_j(A_i)=0 \quad i=1,...,m \tag{6}$$

where $\lambda_j\geq 0$ and $\lambda_jf_j=0$. Equation (6) is the optimality criterion. In the case of the displacement constraint problem, Eq. (4) can be written as

$$f_j(A_i)=\sum_{i=1}^{m}\frac{E_{ij}}{A_i}-\bar{C}_j\leq 0 \quad j=1,...,p \tag{7}$$

where E_{ij} is the flexibility coefficient given by

$$E_{ij}=\{r\}_i^t[k]_i\{s^j\}_iA_i \tag{8}$$

where $\{r\}_i$ and $\{s^j\}_i$ are the displacement vectors associated with the ith element due to the applied load vector $\{R\}$ and the virtual load vector $\{S^j\}$ corresponding to the jth constraint. For the bar structure

$$E_{ij}=F_iU_i^j\ell_i/E_i \tag{9}$$

where F_i is the force in the ith bar due to the applied load, U_i^j is the force in the ith bar due to the virtual load vector $\{S^j\}$, and E_i is the elastic modulus of the ith bar. The coefficients E_{ij} are constant for statically determinate structures, and for indeterminate structures they depend on the design variables A_i. However, they may be assumed to be constant for small changes in A_i.

Using Eqs. (6) and (7), the optimality condition can be written as

$$1=\sum_{j=1}^{p}\lambda_j\frac{E_{ij}}{\rho_i\ell_iA_i^2} \quad i=1,...,m \tag{10}$$

where $\lambda_j\geq 0$ and $\lambda_jf_j=0$.

The optimum structure has to satisfy Eq. (10) and the constraint, Eq. (7). These are $(m+p)$ nonlinear equations corresponding to the m design variables A_i and the p Lagrange multipliers λ_j. For the bar structures, the allowable stress constraint in each bar can be replaced by the allowable relative displacement constraint between the nodes connecting the bar, and the problem can be treated as a multiple displacement constraint problem.

III. Methods of Solution

This section discusses various schemes available to solve Eqs. (7) and (10). Since these equations are nonlinear, the solution schemes are iterative in nature and are based on the use of recurrence relations. The optimality criterion is used to derive a relation to modify the design variables A_i. The constraint equations are used to obtain relations for the evaluation of the Lagrange parameters. When there is only one constraint on the structure, the single Lagrange multiplier can be explicitly defined and can be used to derive the recurrence relation for the design variable.[3,4] When there is more than one constraint, however, the Lagrange multipliers must be evaluated by using an iterative method.

The available solution schemes will be derived in this section with common notation to facilitate their comparison in the next section. In all relations the subscript ν refers to the iteration number. The index i refers to the element number and it goes from 1 to m. The index j refers to the constraint and it goes from 1 to p.

A. Newton-Raphson Method

First, we consider the direct solution of Eqs. (10) and (7) by the Newton-Raphson method. The equations to be solved are the optimality conditions

$$A_i^2=\sum_{j=1}^{p}\lambda_j\frac{E_{ij}}{\rho_i\ell_i} \tag{11}$$

and the constraints

$$f_j=\sum_{i=1}^{m}\frac{E_{ij}}{A_i}-\bar{C}_j \tag{12}$$

Equation (12) is nonlinear in the Lagrange multipliers. The increment Δf_j in the jth constraint can be written as

$$\Delta f_j=f_j(\lambda+\Delta\lambda)-f_j(\lambda)=\sum_{k=1}^{p}\frac{\partial f_j}{\partial\lambda_k}\Delta\lambda_k \tag{13}$$

where $\lambda=(\lambda_1,...,\lambda_p)$ and $\Delta\lambda=(\Delta\lambda_1,...,\Delta\lambda_p)$. Since in the Newton-Raphson method the increment $\Delta\lambda$ is selected so as to satisfy the equation $f_j(\lambda+\Delta\lambda)=0$, the iteration relation can be written as

$$-\eta\{f\}_\nu=[H]_\nu\{\lambda^{\nu+1}-\lambda^\nu\} \tag{14}$$

or

$$\{\lambda\}^{\nu+1}=\{\lambda\}^\nu-\eta[H]_\nu^{-1}\{f\}_\nu \tag{15}$$

where $(\nu+1)$ and ν are the iteration numbers, η is a parameter introduced to control the step size, and $[H]^{-1}$ is the inverse of the Hessian matrix $[H]$ whose elements are given by

$$H_{jk}=\partial f_j/\partial\lambda_k \tag{16}$$

Differentiating Eq. (12) and remembering that A_i is a function of λ_j, as defined by Eq. (11), one obtains

$$\frac{\partial f_j}{\partial\lambda_k}=-\frac{1}{2}\sum_{i=1}^{m}\frac{E_{ij}E_{ik}}{\rho_i\ell_iA_i^3} \tag{17}$$

When the elements are separated into active and passive elements, the summation in Eq. (17) is taken over the active elements only. The sizes of the passive elements are governed by the minimum size or some other criteria. In using Eq. (15), only active constraints corresponding to the positive Lagrange multipliers must be considered.

The iterative process in this method consists of using Eqs. (11) and (15) alternately until the constraint equations are satisfied. After the constraints are satisfied, the coefficients E_{ij} are updated by reanalyzing the structure. In using Eq. (15), it is necessary to select proper initial values of the Lagrange multipliers and the convergence of the method depends on this selection. When the number of Lagrange multipliers is large and when the active constraints change from one iteration to the next, it is generally very difficult to assign proper initial values. Reference 7 uses this approach with $\eta=1$. In Ref. 20, the Lagrange multipliers are determined by maximizing Eq. (5) expressed as a function of λ's (dual theorem) with constraints $\lambda_j \geq 0$. When only equality constraints are considered, maximizing Eq. (5) with respect to λ's is equivalent to satisfying the constraint equations [Eq. (4)].

B. Recurrence Relations for the Design Variables

The recurrence relation for the design variable A_i can be written by using Eq. (10). Multiplying both sides of this equation by $(A_i)^n$ and taking the nth root gives

$$A_i = A_i\left(\sum_{j=1}^{p} \lambda_j \frac{E_{ij}}{\rho_i \ell_i A_i^2}\right)^{1/n} \tag{18}$$

Based on this equation, a recurrence relation for the design variable can be written as

$$A_i^{\nu+1} = A_i^{\nu}\left(\sum_{j=1}^{p} \lambda_j \frac{E_{ij}}{\rho_i \ell_i A_i^2}\right)_{\nu}^{1/n} \tag{19}$$

where$(\nu+1)$ and ν are the iteration numbers, and the parameter n determines the step size. Equation (19) can be called an exponential recurrence relation. Equation (18) is identical to Eq. (11) for $n=2$. In Refs. 4, 6, 11, and 12, Eq. (19) is used with $n=2$, and each time the areas are modified, the structure is reanalyzed to update the coefficients E_{ij}.

A linear form of a recurrence relation, similar to the one suggested in Ref. 15, can be derived as follows. Multiplying both sides of Eq. (10) by $(1-\alpha)A_i$ and rearranging gives

$$A_i = A_i\left(\alpha + (1-\alpha)\sum_{j=1}^{p} \lambda_j \frac{E_{ij}}{\rho_i \ell_i A_i^2}\right) \tag{20}$$

where α is a parameter which controls the step size. Since A_i appears on both sides of this equation, a recurrence relation can be written as

$$A_i^{\nu+1} = A_i^{\nu}\left(\alpha + (1-\alpha)\sum_{j=1}^{p} \lambda_j \frac{E_{ij}}{\rho_i \ell_i A_i^2}\right)_{\nu} \tag{21}$$

Equation (21) is used in Refs. 16-18, and the structure is reanalyzed each time after the design variables are modified.

In order to use Eq. (19) or (21), it is necessary for the Lagrange multipliers to be known. In the next section, two schemes to evaluate the Lagrange multipliers are discussed.

C. Recurrence Relations for the Lagrange Multipliers

The recurrence relations for the Lagrange multipliers can be formulated by using the constraint equations. If all the constraints are assumed to be equality constraints, Eq. (4) can be written as

$$C_j = \bar{C}_j \tag{22}$$

Multiplying both sides of this equation by $(\lambda_j)^b$ and taking the bth root gives

$$\lambda_j = \lambda_j\left(\frac{C_j}{\bar{C}_j}\right)^{1/b} \tag{23}$$

This equation can be written as a recurrence relation in the form

$$\lambda_j^{\nu+1} = \lambda_j^{\nu}\left(\frac{C_j^{\nu}}{\bar{C}_j}\right)^{1/b} \tag{24}$$

where $(\nu+1)$ and ν are the iteration numbers. In using Eq. (24) it is essential to assume the values of the Lagrange multipliers for the first iteration. In Eq. (24) the parameter b determines the step size. This equation and Eq. (19) for the design variables are used in Refs. 8, 11, 12, 14, and 18.

A set of linear equations similar to the one suggested in Ref. 15 can be obtained by considering the change in the constraint f_j due to the change in the design variable A_i. The change in the jth constraint Δf_j can be written as

$$\Delta f_j = f_j(A+\Delta A) - f_j(A) = \sum_{i=1}^{m} \frac{\partial f_j}{\partial A_i}\Delta A_i \tag{25}$$

where $A=(A_1,A_2,\ldots,A_m)$ and $\Delta A = (\Delta A_1, \Delta A_2, \ldots, \Delta A_m)$. Since the increment ΔA is selected so that $f_j(A+\Delta A)=0$, using Eq. (4), Eq. (25) can be written as

$$\bar{C}_j - C_j^{\nu} = \sum_{i=1}^{m} \frac{\partial f_j}{\partial A_i}(A_i^{\nu+1} - A_i^{\nu}) \tag{26}$$

where $(\nu+1)$ and ν are the iteration numbers. Differentiating Eq. (7) with respect to A_i gives

$$\frac{\partial f_j}{\partial A_i} = -\frac{E_{ij}}{A_i^2} \tag{27}$$

Using Eq. (21), the increment in areas is given by

$$\Delta A_i^{\nu} = (\alpha-1)\left(1 - \sum_{j=1}^{p} \lambda_j \frac{E_{ij}}{\rho_i \ell_i A_i^2}\right)_{\nu} A_i^{\nu} \tag{28}$$

Substituting Eqs. (27) and (28) in Eq. (26), setting $\lambda_k = \lambda_k^{\nu+1}$ and rearranging the terms gives

$$\sum_{k=1}^{p} \lambda_k^{\nu+1} \sum_{i=1}^{m}\left(\frac{E_{ij}E_{ik}}{\rho_i \ell_i A_i^3}\right)_{\nu} = \frac{C_j^{\nu}(2-\alpha) - \bar{C}_j}{(1-\alpha)} \tag{29}$$

If the elements are separated into active and passive elements, Eq. (29) takes the form

$$\sum_{k=1}^{p} \lambda_k^{\nu+1} \sum_{i=1}^{m_1}\left(\frac{E_{ij}E_{ik}}{\rho_i \ell_i A_i^3}\right)_{\nu} = \sum_{i=1}^{m_1}\left(\frac{E_{ij}}{A_i}\right)_{\nu} - \frac{1}{(1-\alpha)}\left[(\bar{C}_j - C_j) + \sum_{i=m_1+1}^{m} \frac{E_{ij}}{A_i^2}\Delta A_i^{\nu}(P)\right]_{\nu} \tag{30}$$

where m_1 is the number of active elements and $\Delta A_i^{\nu}(P) = (A_i^* - A_i^{\nu})$. A_i^* is the size dictated by the minimum requirement. Generally, in an iterative scheme if any element reaches minimum size, it remains minimum throughout the subsequent iterations and $\Delta A_i^{\nu}(P)=0$.

Equation (30), in conjunction with Eq. (21), is used in Refs. 16-18. The structure is reanalyzed each time after finding the new design vector.

IV. Comparison of Different Recurrence Relations

In this section the recurrence relations derived in the last section are compared, some new relations are given, and a new approach is proposed. For simplicity in this section all elements are assumed to be active. However, if the elements are divided into active and passive elements, necessary modifications to the equations in this section can be made similar to Eq. (30).

In the last section, the exponential recurrence relation [Eq. (19)] and the linear recurrence relation [Eq. (21)] were derived to modify the design variables. Even though both relations have different forms, they can be shown to be equivalent. Equation (19) can be rewritten as

$$A_i^{\nu+1} = A_i^{\nu}\left[1 + \left(\sum_{j=1}^{p} \lambda_j \frac{E_{ij}}{\rho_i \ell_i A_i^2} - 1\right)\right]_{\nu}^{1/n} \tag{31}$$

Since, near the optimum

$$\sum_{j=1}^{p} \lambda_j \frac{E_{ij}}{\rho_i \ell_i A_i^2}$$

is nearly equal to unity,

$$\sum_{j=1}^{p}\lambda_j\frac{E_{ij}}{\rho_i\ell_iA_i^2}-1$$

is small compared to unity. Hence, if Eq. (31) is expanded by the binominal theorem and only the linear terms are retained, Eq. (31) can be approximated as

$$A_i^{\nu+1}=A_i^\nu\left[1+\frac{1}{n}\left(\sum_{j=1}^{p}\lambda_j\frac{E_{ij}}{\rho_i\ell_iA_i^2}-1\right)\right]_\nu \tag{32}$$

or

$$A_i^{\nu+1}=A_i^\nu\left(\alpha+(1-\alpha)\sum_{j=1}^{p}\lambda_j\frac{E_{ij}}{\rho_i\ell_iA_i^2}\right)_\nu \tag{33}$$

where

$$\alpha=[1-(1/n)] \tag{34}$$

Equation (33) is the same as Eq. (21) which was derived by a different approach. Equation (33) can be considered as a linear form of Eq. (19). During iteration, Eq. (19) creates a new design by modifying each component of the old one by a factor, while Eq. (21) or (33) creates a new design vector as the weighted sum of the old vector and the new one created by Eq. (19) with $n=1$.

The recurrence relation for the Lagrange multipliers given in Eq. (24) can also be linearized by using an argument similar to the one used to derive Eq. (32). The linear form of Eq. (24) can be written as

$$\lambda_j^{\nu+1}=\lambda_j^\nu\left(\frac{b+1}{b}-\frac{1}{b}\frac{\bar{C}_j}{C_j^\nu}\right) \tag{35}$$

Equation (35) can also be written as

$$\bar{C}_j-C_j^\nu=b\left(C_j^\nu-\frac{\lambda_j^{\nu+1}}{\lambda_j^\nu}\right) \tag{36}$$

Equations (35) or (36) can also be used in the same way as Eq. (24) to modify the initially assumed Lagrange parameters.

In Sec. III, the application of the Newton-Raphson iterative technique was discussed to solve the optimality criterion and the constraint equations. Now it will be shown that the iteration relations used in this method can be written in a more convenient form. Equation (14) or (15) can be written as

$$[H]_\nu\{\lambda\}^{\nu+1}=[H]_\nu\{\lambda\}^\nu-\eta\{f\}_\nu \tag{37}$$

Using Eq. (17), Eq. (37) can be written as

$$\sum_{k=1}^{p}\lambda_k^{\nu+1}\sum_{i=1}^{m}\left(\frac{E_{ij}E_{ik}}{\rho_i\ell_iA_i^3}\right)_\nu=\sum_{k=1}^{p}\lambda_k^\nu\sum_{i=1}^{m}\left(\frac{E_{ij}E_{ik}}{\rho_i\ell_iA_i^3}\right)_\nu+2\eta f_j^\nu \tag{38}$$

Substituting A_i^2 from Eq. (11) in Eq. (12) and rearranging, one obtains

$$f_j^\nu=\sum_{k=1}^{p}\lambda_k^\nu\sum_{i=1}^{m}\left(\frac{E_{ij}E_{ik}}{\rho_i\ell_iA_i^3}\right)_\nu-\bar{C}_j \tag{39}$$

Using Eq. (39) and remembering that $f_j^\nu=(C_j^\nu-\bar{C}_j)$, Eq. (38) can be written as:

$$\sum_{k=1}^{p}\lambda_k^{\nu+1}\sum_{i=1}^{m}\left(\frac{E_{ij}E_{ik}}{\rho_i\ell_iA_i^3}\right)_\nu=(2\eta+1)C_j^\nu-2\eta\bar{C}_j \tag{40}$$

When η is equal to unity, this equation becomes

$$\sum_{k=1}^{p}\lambda_k^{\nu+1}\sum_{i=1}^{m}\left(\frac{E_{ij}E_{ik}}{\rho_i\ell_iA_i^3}\right)_\nu=3C_j^\nu-2\bar{C}_j \tag{41}$$

The main advantage of writing Eq. (15) in the form as given by Eq. (40) is that it is not necessary to assume the initial values of the Lagrange multipliers to start the Newton-Raphson iterative scheme. Equation (40) can be solved at each step to obtain the new values of λ_j, after modifying the design variables by using Eq. (11). It may be pointed out here that the convergence of the Newton-Raphson technique depends upon initially selected values of the unknown variables, and generally, it is very difficult to assign the initial values when the number of unknowns is large.

In Sec. III, Eq. (29) was obtained by considering the change in the constraint Δf_j due to the change in the design variable ΔA_i. Using Eq. (34), Eq. (29) can be written as:

$$\sum_{k=1}^{p}\lambda_k^{\nu+1}\sum_{i=1}^{m}\left(\frac{E_{ij}E_{ik}}{\rho_i\ell_iA_i^3}\right)_\nu=C_j^\nu(n+1)-n\bar{C}_j \tag{42}$$

If in Eq. (42) the parameter n which controls the step size in the recurrence relation for the design variable A_i is set to equal to 2, then this equation becomes

$$\sum_{k=1}^{p}\lambda_k^{\nu+1}\sum_{i=1}^{m}\left(\frac{E_{ij}E_{ik}}{\rho_i\ell_iA_i^3}\right)_\nu=2C_j^\nu-2\bar{C}_j \tag{43}$$

It is interesting to see that Eq. (41), which is derived from the Newton-Raphson iteration relation [Eq. (15)] is the same as Eq. (43). The equivalence of Eq. (41) and Eq. (29) indicates that the method proposed in Ref. 15 is equivalent to the Newton-Raphson technique, if, in the Newton-Raphson method, only one iteration is completed and the structure is reanalyzed. Normally in the Newton-Raphson method the structure is analyzed after the constraints are satisfied or after a specified number of iterations. In the method proposed in Ref. 15, it is not required to assume initial values of the Lagrange multipliers.

This similarity between the two methods is evident, if it is observed that if in Eq. (13)

$$\frac{\partial f_j}{\partial\lambda_k}=\sum_{i=1}^{m}\frac{\partial f_j}{\partial A_i}\frac{\partial A_i}{\partial\lambda_k} \tag{44}$$

and in Eq. (25)

$$\Delta A_i=\sum_{k=1}^{p}\frac{\partial A_i}{\partial\lambda_k}\Delta\lambda_k \tag{45}$$

are substituted, both equations reduce to the same equation.

It will now be shown that Eq. (35), which is the linearized form of Eq. (24), is an approximation of Eq. (29). If in Eq. (29), the off-diagonal terms in the matrix multiplying the vector of the Lagrange multipliers λ_j are neglected, then Eq. (29) can be approximated as:

$$\lambda_j^{\nu+1}\sum_{i=1}^{m}\left(\frac{E_{ij}E_{ij}}{\rho_i\ell_iA_i^3}\right)_\nu=\frac{C_j^\nu(2-\alpha)-\bar{C}_j}{(1-\alpha)} \tag{46}$$

These are uncoupled equations and assume that the constraints are independent of one another. This means that while considering the jth constraint, the contribution of all other constraints may be neglected. With this assumption, using Eq. (10) the term $E_{ij}/(\rho_i\ell_iA_i^2)$ can be set equal to $1/\lambda_j^\nu$. Furthermore, since

$$\sum_{i=1}^{m}E_{ij}/A_i^\nu=C_j^\nu$$

Equation (46) can be written as

$$\frac{\lambda_j^{\nu+1}}{\lambda_j^\nu} C_j^\nu = \frac{C_j^\nu(2-\alpha) - \bar{C}_j}{(1-\alpha)} \tag{47}$$

or

$$\lambda_j^{\nu+1} = \lambda_j^\nu \left(\frac{2-\alpha}{1-\alpha} - \frac{1}{1-\alpha} \frac{\bar{C}_j}{C_j^\nu} \right) \tag{48}$$

Comparing Eq. (35) and Eq. (48) it is seen that they have the same form and would be identical for

$$1/b = 1/(1-\alpha) \tag{49}$$

Using the same argument as used to write Eq. (48) from Eq. (29), Eq. (40) can be approximated as

$$\lambda_j^{\nu+1} = \lambda_j^\nu \left[(2\eta+1) - 2\eta \frac{\bar{C}_j}{C_j^\nu} \right] \tag{50}$$

Comparing Eqs. (35) and (50), it is seen that they are identical for

$$1/b = 2\eta \tag{51}$$

For $\alpha = 1/2$ and $\eta = 1$, Eqs. (48) and (50) are identical.

At this point it will be shown that the generalized forms of Eq. (40) and Eq. (29) are equivalent. The coefficients H_{ij} in Eq. (15) are derived by using Eq. (11). When $n = 2$, Eqs. (18) and (11) are equivalent. The generalized form of Eq. (40) would be

$$\sum_{k=1}^{p} \lambda_k^{\nu+1} \sum_{i=1}^{m} \left(\frac{E_{ij}E_{ik}}{\rho_i \ell_i A_i^3} \right)_\nu = (n\eta+1) C_j^\nu - n\eta \bar{C}_j \tag{52}$$

In Sec. III Eq. (29) was derived by equating Δf_j to $\bar{C}_j - C_j^\nu$; however, if Δf_j is set equal to $\eta\ (\bar{C}_j - C_j^\nu)$ where η is a parameter introduced to control the step size towards the constraint surface, the generalized form of Eq. (29) would be

$$\sum_{k=1}^{p} \lambda_k^{\nu+1} \sum_{i=1}^{m} \left(\frac{E_{ij}E_{ik}}{\rho_i \ell_i A_i^3} \right)_\nu = \frac{C_j^\nu(1-\alpha+\eta) - \eta\bar{C}_j}{(1-\alpha)} \tag{53}$$

If the parameter α in this equation is set equal to $(1 - 1/n)$ [Eq. (34)], then Eq. (53) is identical to Eq. (52).

When all of the constraints are satisfied as equality constraints, i.e., $f_j^\nu = C_j^\nu - \bar{C}_j = 0$, then Eqs. (40, 41, 52, and 53) become

$$\sum_{k=1}^{p} \lambda_k^{\nu+1} \sum_{i=1}^{m} \left(\frac{E_{ij}E_{jk}}{\rho_i \ell_i A_i^3} \right)_\nu = \bar{C}_j \tag{54}$$

Equation (54) is valid at the optimum for all the active constraints with m equal to the number of active elements.

If in Eq. (53), the off-diagonal terms in the matrix multiplying the vector of the Lagrange multipliers are neglected, then this equation can be approximated as

$$\lambda_j^{\nu+1} \sum_{i=1}^{m} \left(\frac{E_{ij}E_{ij}}{\rho_i \ell_i A_i^3} \right)_\nu = \frac{C_j^\nu(1-\alpha+\eta) - \eta\bar{C}_j}{(1-\alpha)} \tag{55}$$

If in Eq. (55), C_j^ν is set equal to $\bar{C}_j$, the limiting value of the constraint, then one obtains

$$\lambda_j^{\nu+1} \sum_{i=1}^{m} \left(\frac{E_{ij}E_{ij}}{\rho_i \ell_i A_i^3} \right)_\nu = \bar{C}_j \tag{56}$$

In the case of a bar structure where the allowable stresses are not the same, the minimum weight design can be obtained by replacing the variable stress constraints by an equivalent displacement constraint problem. If in the bar structure it is assumed that there are no coupling effects, and the virtual energy in an element is only due to the unit load in that element (which is true for a statically determinate structure), then Eq. (56) becomes

$$\lambda_i^{\nu+1} \left(\frac{E_{ii}E_{ii}}{\rho_i \ell_i A_i^3} \right)_\nu = \bar{C}_i \quad i = 1,\ldots,m \tag{57}$$

where

$$\bar{C}_i = \frac{\bar{\sigma}_i}{E_i} \ell_i \tag{58}$$

In Eq. (58), $\bar{\sigma}_i$ is the maximum allowable stress in the ith element. The use of Eq. (57) to design a minimum weight structure was proposed in Ref. 14, and its convergence behavior with Eq. (24) is also discussed.

Now, to summarize the results, the design variables A_i can be modified by using the exponential form, Eq. (19) or the linear form, Eq. (33). In these two equations, the step size is controlled by proper selection of the parameters n and α. These two parameters are related by Eq. (34).

The Lagrange multipliers can be evaluated by using Eq. (24) in the exponential form or the linear relations given in Eqs. (35) and (36). The step size is controlled by the parameter b. In order to use these equations, it is necessary to assume the initial values but it is not necessary to separate the constraints into active and passive categories. In using Eq. (24), the λ's corresponding to the passive constraints become small and thus their effect is negligible. When there are displacement and stress constraints, it is found that this method gives the best results when all the constraints are included in the problem. If only displacement constraints are considered, the use of these equations may not converge to the minimum weight design. Note that with this procedure it is not required to solve a set of linear equations to evaluate the Lagrange multipliers. Generally, all initial values of the multipliers are assumed to be the same; however, it is found that for bar structures, with stress constraints, it is advantageous to assign them so that they are proportional to the forces in the bars. If the Lagrangian multipliers are assumed to be proportional to the forces in the bars for all iterations, Eq. (10) reduces to the constant strain energy density criterion proposed in Refs. 1, 2, and 6. This transition is discussed in Ref. 14.

In the Newton-Raphson procedure, as given by Eq. (15), generally there is a difficulty with selecting initial values for the Lagrange multipliers. This iterative procedure also does not eliminate passive constraints automatically.

To use Eq. (29), it is necessary to solve for each iteration a set of linear equations to evaluate the Lagrange multipliers, and when the number of constraints is large, this is time consuming. This procedure also does not automatically eliminate the passive constraints unless a special solution scheme is used.[16]

The approximate forms of the recurrence relations, as given by Eqs. (46, 55, 56, or 57), can also be used to evaluate the multipliers. However, their successful use may depend on the nature of the problem. All the relations generally behave well for the first few iterations; however, when the interative procedure approaches the minimum weight design, the approximate nature of the relations may cause the iterative procedure to oscillate, to diverge, or to slow down.

For any of the procedures discussed in this section, the convergence behavior is also strongly influenced by the values chosen for their "step size" parameters n, α, b, and η. The best values are usually also problem-dependent.

It has been shown that Eq. (15) used in the Newton-Raphson method can be reduced to Eq. (40) or to its generalized form, Eq. (52). If, instead of using Eq. (15) in the Newton-Raphson procedure Eq. (52) is used, then it is not necessary to assume initial values for the Lagrange parameters. They can be evaluated during each iteration by solving a set of linear equations. The use of this approach is illustrated in Sec. V. Since Eqs. (24, 35, and 36) are approximations to Eqs. (52), one may use these equations in its place. In Sec. V, some problems are solved using such approximations, and it appears that they might provide the most efficient approach.

V. Illustrative Problems

In this section the behavior of a few selected procedures is demonstrated. The relations used to modify the design variables are either Eq. (19) or Eq. (21). The Lagrange multipliers are evaluated by using either Eq. (24) or Eq. (29). The Gauss-Seidel iterative solution scheme is used to solve Eq. (29) and to eliminate passive constraints.[16] The problems are solved either by analyzing the structure after each iteration, i.e., modifying the design variables and evaluating the Lagrange multipliers, or by analyzing the structure after more than one iteration as suggested at the end of Sec. IV. This latter approach may be viewed as a modified Newton-Raphson technique. All the structures are required to satisfy the stress constraints in all the elements in addition to the imposed displacement constraints. For all the problems, the initial sizes of all the elements are equal in the first iteration. The weight of the structure at any iteration is the weight of the scaled design satisfying all the constraints.

Example I

The 10-bar truss, shown in Fig. 1, is designed to satisfy the stress and displacement constraints. The stress limit in all the bars is 25 ksi and the displacement limit of ±2.0 in. is imposed at all the node points.

Case I

The 10-bar truss problem is solved by considering only two displacement limits as the constraints in Eq. (7). The two constraints considered are the vertical displacements at nodes 1 and 2. The stress limits are considered simply through scaling. For all the problems discussed in Case I, the following equivalent parameters were selected: $n=4$ [Eq. (19)]; $\alpha=0.75$ [Eq. (21)]; $b=0.25$ [Eq. (24)]; [see Eqs. (34) and (49)]. The iteration history for Case I is shown in Fig. 2.

Case Ia

Equations (19) and (24) are used to iterate the design variables and to determine the Lagrange multipliers, respectively. The initial relative values of the Lagrange multipliers are all 1.0. The structure is reanalyzed after each iteration. A design with a minimum weight of 5187.84 lb was obtained after 12 analyses (see Fig. 2). The weight of the structure did not improve with additional iterations.

Case Ib

The areas are modified by using Eq. (21) and the Lagrange multipliers are evaluated by using Eq. (29). The constraints are considered as inequality constraints, i.e., negative Lagrange multiplies are not permissible. The structure is analyzed after each iteration. The minimum weight design was obtained after 28 analyses with a weight of 5076.66 lb and it remained unchanged with additional iterations (see Fig. 2).

Case Ic

This case is the same as Case Ib except that a maximum of 50 iterations were allowed before reanalyzing the structure (modified Newton-Raphson). The criteria set to satisfy the constraints [Eq. (7)] is 10^{-7}. Except for the first few analyses, not more than 10 iterations were needed to satisfy the criteria.

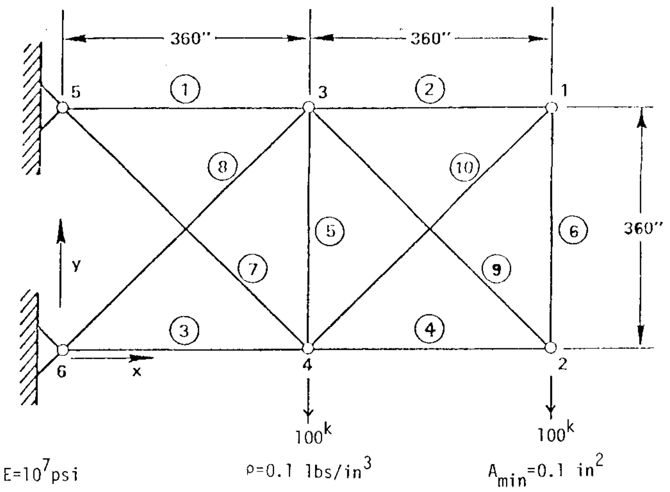

Fig. 1 Ten-bar truss.

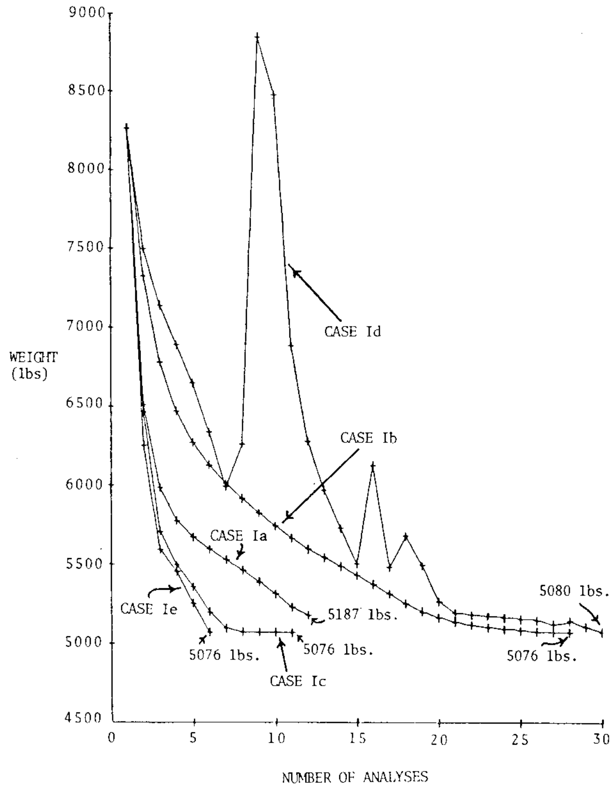

Fig. 2 Iteration history for 10-bar truss (Case I).

A design with a weight of 5079.24 lb was obtained after 8 analyses and 5076.66 lb after 11 analyses (see Fig. 2).

Case Id

This case is the same as Case Ib except that the constraints are assumed to be equality constraints, i.e., negative Lagrange multipliers are permissible during the iterations. The iteration history is shown in Fig. 2. A design with a weight of 5076.66 lb was obtained after 35 analyses.

Case Ie

This case is the same as Case Ic except that the constraints were assumed to be equality constraints. The iteration history is shown in Fig. 2. A design with a weight of 5076.84 lb was obtained after only six analyses.

For Case I, at the optimum, the vertical displacements at nodes 1 and 2 are equal to 2.0-in. In all members the stresses are less than the maximum allowable. The maximum stress is

Table 1 Minimum weight designs of 10-bar truss

Member	Case I	Case II	Case III
1	30.7297	30.5210	7.9000
2	0.1000	0.1000	0.1000
3	23.9407	23.1999	8.1000
4	14.7331	15.2229	3.9000
5	0.1000	0.1000	0.1000
6	0.1000	0.5514	0.1000
7	8.3406	7.4572	5.7983
8	20.9510	21.0364	5.5154
9	20.8358	21.5284	3.6770
10	0.1000	0.1000	0.1414
Weight	5076.66	5060.85	1497.60

equal to 20.36 ksi in member 5. The areas of the elements are given in Table 1.

Case II

This problem is the same as Case I except that the stress constraints inthe bars were included in the constraint equations [Eq. (7)]. The allowable stress constraint in each bar was replaced by the allowable relative displacement constraint between the two nodes connecting the bar. Thus, there were a total of 12 constraints on the structure (10 stress constraints and 2 displacement constraints). For Case II, $n=2$, $\alpha=0.5$, and $b=0.5$. After each reanalysis, all constraints were again considered active initially. The iteration history for Case II is shown in Figs. 3 and 4.

Case IIa

This case is the same as Case Ia except that all the above 12 constraints were considered in Eq. (7). The iteration history is shown in Fig. 3. The weight of the structure between iterations 11-17 was found to be greater than 9000 lb, hence it is not shown on the figure. A design with a weight of 5062.50 lb was obtained at the 19th analysis. After the 30th analysis, the weight of the structure was 5074.13 lb. After this the convergence was found to be very slow until the weight becomes 5060.85 lb.

Table 2 Iteration history

Analysis number	Case Ic	Case Ie	Case IId	Case IIId
1	8266.15	8266.15	8266.15	3434.97
2	6464.05	6255.54	5993.52	1733.43
3	5713.37	5598.99	5815.76	1645.71
4	5499.51	5457.51	5690.67	1640.95
5	5363.67	5259.19	5574.27	1534.64
6	5206.57	5076.84	5447.55	1527.13
7	5101.26		5304.99	1521.54
8	5079.89		5198.85	1516.67
9	5079.24		5105.80	1512.38
10	5078.24		5075.20	1508.68
11	5076.66		5065.18	1505.49
12			5061.31	1502.79
13			5060.87	1500.34
14			5060.85	1497.60

Case IIb

This case is the same as Case IIa except that a maximum of 50 iterations were allowed before renalyzing the structure. (modified Newton-Raphson). The iteration history is shown in Fig. 3. A design with a weight of 5060.88 lb was obtained with 15 analyses. The weight of the structure at the 11th and 12th analysis is greater than 9000 lb, hence it is not shown in the figure.

Case IIc

This case is the same as Case Ib except that there were 12 constraints on the structure. The iteration history is shown in Fig. 4. A minimum weight design was obtained after 20 analyses. The weight of the structure did not improve beyond 5076.66 lb.

Case IId

This case is the same as Case Ic. A design with a weight of 5060.85 lb was obtained after 14 analyses. It is of interest to see that in this case a design with a weight of 5060.85 lb was obtained instead of 5076.66 lb as in Case IIc. The difference

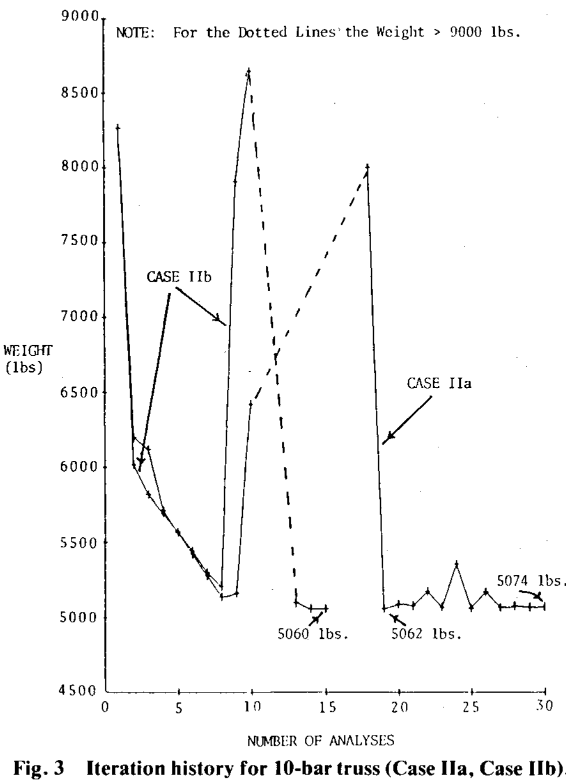

Fig. 3 Iteration history for 10-bar truss (Case IIa, Case IIb).

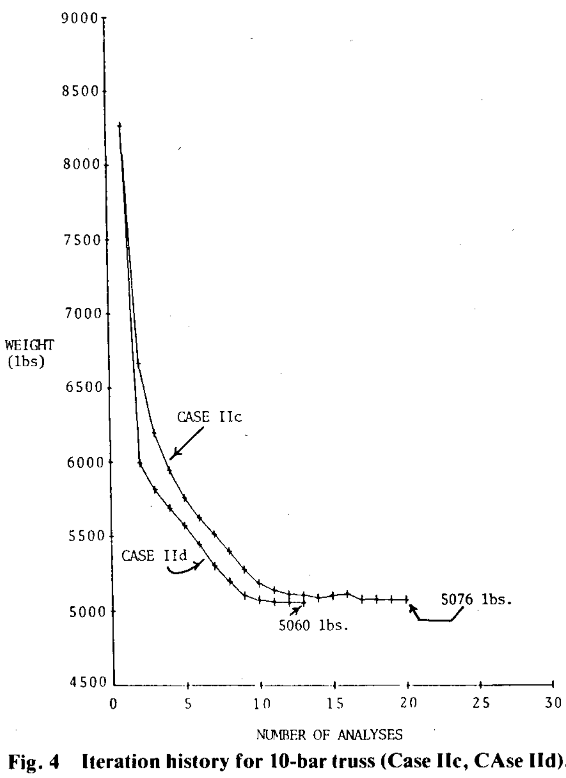

Fig. 4 Iteration history for 10-bar truss (Case IIc, CAse IId).

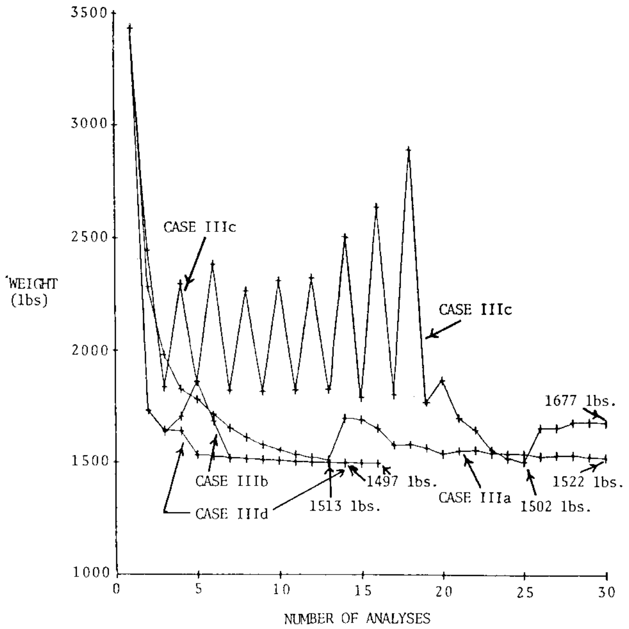

Fig. 5 Iteration history for 10-bar truss (Case III).

between Case IIc and IId is that in Case IId more than one iteration was performed before reanalyzing the structure.

For the design with a weight of 5060.85 lb, the vertical deflections at nodes 1 and 2 were 2.000 and 1.9914 in., respectively. The stresses in all the elements were less than the maximum allowable except for bar 5. The stress in bar 5 was 25 ksi and the area was 0.1 in.^{2}, which was also the minimum allowable. The areas of the elements are given in Table 1.

Case III

The 10-bar truss was designed with no displacement constraints, but the allowable stress in bar 9 was increased to 50 ksi. The problem was solved with 10 stress constraints converted into equivalent displacement constraints. For this case, $n=2$, $\alpha=0.5$, and $b=n$. After each analysis, all constraints were again considered active initially.

Case IIIa

This case is the same as Case IIa. The iteration history is shown in Fig. 5. A design with a weight of 1513.61 lb was obtained at the 13th analysis. The weight of the structure after the 30th analysis was 1522.88 lb. After this the convergence was very slow, until the weight of the structure became 1497.60 lb.

Case IIIb

This case is the same as Case IIb. In this case, during the iterations, the parameter n was increased by a factor of 1.25 whenever the weight of the structure was greater than the previous one. A design with a weight of 1497.60 lb was obtained with 16 analyses of the structure. The iteration history is shown in Fig. 5.

Case IIIc

This case is the same as Case IIc. The iteration history is shown in Fig. 5. The weight of the structure was found to oscillate after the 3rd analysis. A design with a weight of 1502.82 lb was obtained at the 25th analysis. After this, the weight was found to increase to 1677.64 lb and remain constant. The erratic behavior in this case seems to be due to $\alpha=0.5$ being too large a step size for this algorithm.

Case IIId

This case is the same as Case IId. In this case, the step size was reduced in the same proportion as in Case IIIb, whenever the weight of the structure goes up. A design with a weight of 1497.60 lb was obtained after 14 analyses. The iteration history is shown in Fig. 5.

The areas of the members of the minimum weight design are given in Table 1. For this case, the stress in bar 9 is 37.5 ksi even though the allowable stress is 50 ksi. The reason is that bar 9 cannot be stressed beyond 37.5 ksi without overstressing the adjacent bars beyond their allowable of 25.0 ksi.

The iterative history for Cases Ic, Ie, IId, and IIId are given in Table 2.

VI. Summary and Conclusion

This paper has attempted in Sec. IV to show the relations between different algorithms proposed by several investigators to design a minimum weight structure by using an optimality criterion. Even though the paper addresses itself to the problem of displacement constraints only, the relations and conclusions are applicable to other types of constraints. In all these methods, the objective is to directly satisfy the optimality criterion and the constraint equations by an iterative technique, thereby indirectly optimizing the structure. The design variables are modified by using recurrence relations derived from the optimality criterion. For all the methods, the recurrence relations used to modify the design variables were shown to be equivalent to the exponential or linearized form. In using the recurrence relation, the step size is controlled by a parameter which may be fixed or varied. The relations used to solve for the Lagrange multipliers in the various methods are obtained from the constraint equations by making various degrees of approximations. The lesser the approximations, the more complicated are the relations requiring more computational time for solution and the separation of the constraints into active and passive categories. This may be a governing factor in their use when the number of constraints on the structure is large. The simpler relations need less computational time; however, if the coupling between the different constraints is too strong, their use may not allow the iterative procedure to converge to the minimum weight design without a large number of iterations. The method similar to the Newton-Raphson iterative technique proposed in this paper is found to give the minimum weight design with the least number of analyses. The success of this method depends on the degree of indeterminacy in the structure. If the force distribution does not radically change with the change in the design variables, the numer of iterations can be increased before reanalyzing the structure and the minimum weight design can be obtained with a minimum number of reanalyses.

The convergence characteristics of all the algorithms depend on the selection of a proper step size. At present, this selection is made arbitrarily, or with much experimentation with the same problem. It is necessary to establish certain criteria to select the step size in order to further improve the convergence characteristics of these methods and reduce the number of reanalyses. A more reliable "adaptive" approach than tried in Cases IIIb and IIId should be developed. The sometimes violent oscillations of the scaled weights do not necessarily indicate similar oscillations in the internal force distributions, and may not be the proper quantities to be used to describe convergence behavior or to guide an adaptive scheme. Section V employs only one structure for brevity, the now classical 10-bar truss. A large amount of numerical experimentation was performed utilizing other structures. The general behavior was observed to be similar with these structures and support the results presented here.

References

[1] Venkayya, V. B., Khot, N. S., and Reddy, V. S., "Energy Distribution in an Optimum Structural Design," AFFDL-TR-68-156, March 1969.

[2] Venkayya, V. B., "Design of Optimum Structures," *Journal of Computers and Structures*, Vol. 1, Aug. 1971, pp. 205-209.

[3] Berke, L., "An Efficient Approach to the Minimum Weight Design of Deflection Limited Structures," AFFDL-TM-70-4 FDTR, July 1970.

[4] Gellatly, R. A. and Berke, L., "Optimum Structral Design," AFFDL-TR-70-165, April 1971.

[5] Venkayya, V. B., Khot, N. S., Tischler, V. A., and Taylor, R. F., "Design of Optimum Structures for Dynamic Loads," 3rd Air Force Conference on Matrix Methods in Structural Mechanics, Oct. 1971.

[6] Venkayya, V. B., Khot, N. S., and Berke, L., "Application of Optimality Criteria Approaches to Automated Design of Large Practical Structures," AGARD Second Symposium on Structural Optimization, Milan, Italy, April 1973, pp. 3-1, 3-20.

[7] Taig, I. C. and Kerr, R. I., "Optimization of Aircraft Structures with Multiple Stiffness Requirements," AGARD Second Symposium on Structural Optimization, Milan, Italy, April 1973, pp. 16-1, 16-14.

[8] Nagteggal, J. C., "A New Approach to Optimal Design of Elastic Structures," *Computational Methods in Applied Mechanics and Engineering*, Vol. 2, Feb. 1973, pp. 255-264.

[9] Khot, N. S., Venkayya, V. B., Johnson, C. D., and Tischler, V. A., "Optimization of Fiber Reinforced Composite Structures," *International Journal of Solids and Structures*, Vol. 9, Sept. 1973, pp. 1225-1236.

[10] Venkayya, V. B. and Khot, N. S., "Design of Optimum Structures to Impulse Type Loading," *AIAA Journal*, Vol. 13, Aug. 1975, pp. 989-994.

[11] Berke, L. and Khot, N. S., "Use of Optimality Criteria Methods for Large Scale Systems," AGARD Lecture Series No. 70 on Structural Optimization, Oct. 1974, pp. 1-1, 1-29.

[12] Khot, N. S., Venkayya, V. B., and Berke, L., "Optimum Design of Composite Structures with Stress and Displacement Constraints," *AIAA Journal*, Vol. 14, Feb. 1976, pp. 131-132.

[13] Khot, N. S., Venkayya, V. B., and Berke, L., "Optimum Structural Design with Stability Constraints," *International Journal of Numerical Methods in Engineering*, Vol. 10, Oct. 1976, pp. 1097-1114.

[14] Berke, L. and Khot, N. S., "A Simple Virtual Strain Energy Method to Fully Stress Design Structures with Dissimilar Stress Allowables and Material Properites," AFFDL-TM-77-28-FBR, Dec. 1977.

[15] Kiusalaas, J., "Minimum Weight Design of Structures via Optimality Criteria," NASA TN D-7115, Dec. 1972.

[16] Rizzi, P., "Optimization of Multi-Constrained Structures Based on Optimality Criteria," AIAA/ASME/SAE 17th Structures, Structural Dynamics, and Materials Conference, King of Prussia, Pa., May 1976.

[17] Segenreich, S. A. and McIntosh, S. C., "Weight Optimization Under Multiple Equality Constraints Using an Optimality Criteria," AIAA/ASME/SAE 17th Structures, Structural Dynamics, and Materials Conference, King of Prussia, Pa., May 1976.

[18] Khot, N. S., Venkayya, V. B., and Berke, L., "Experiences with Minimum Weight Design of Structures Using Optimality Criteria Methods," 2nd International Conference on Vehicle Structural Mechanics, Southfield, Mich., April 1977.

[19] Khot, N. S., Berke, L., Venkayya, V. B., and Schrader, K., "Optimum Design of Composite Wing Structures with Twist Constraint for Aeroelastic Tailoring," AFFDL-TR-76-117, Dec. 1976.

[20] Sanders, G. and Fleury, C., "A Mixed Method in Structural Optimization," Paper presented at ASME Energy Technology Conference, Sept. 1977, Houston, Tex.